The Placement of Proposition 1.4 within Euclid's *Elements*

Since at least as far back as Carpus of Antioch, mathematicians and geometers have attempted to justify the placement of Euclid's Prop. 1.4 after Props. 1.1-3. While the other propositions in Euclid's *Elements* (after the first) are dependent on at least one previous proposition, and often several, 1.4 alone seems to appear from nowhere, without having explicit recourse to any of the preceding three propositions. This paper seeks to explain the placement of 1.4 by showing that rigorous geometric proof in fact requires its dependency on the immediately preceding 1.3.

Ancient commentators, finding no overt reference to a dependency on a previous proposition, naturally sought other ways to justify Euclid's placement of 1.4. According to Proclus's commentary (241-243), Carpus of Antioch justified it on the basis of its status as a theorem as opposed to a problem. That is, whereas Props. 1.1-3 pose problems – i.e., *construct* (συντήσασθαι) an equilateral triangle (1.1), *place* (θέσθαι) a straight-line of given length on a point (1.2), *cut* (ἀφελεῖν) a straight-line of given length from a longer straight-line (Prop. 1.3) – Prop. 1.4, as a theorem, proves a truth, i.e., that any two triangles will be equal if they each have two corresponding equal sides around an equal angle. According to Carpus, then, because 1.1-3 are problems, and problems are simpler than theorems, 1.1-3 precedes 1.4. Proclus himself (233-235) justifies the position of 1.4 differently, arguing that because 1.1-3 discover the *possibility* of "triangle" and "equality" as concepts, they must necessarily precede 1.4, which presupposes those concepts.

It is Euclid's unfortunate wording in 1.4 that has, I believe, distracted both ancient and modern scholars from pursuing a logical dependency of 1.4 upon 1.3. In his insisting on the *movement* of figures – one triangle's being "fitted onto" (ἐφαρμοζομένου…ἐπί) another triangle, and a point and line's being placed onto (τιθεμένου…ἐπί) another point and line – Euclid violates a fundamental geometric principle in what looks like a lazy shortcut. As Bertrand Russell (1903, 405) observes of the movement required in 1.4, this "method of superposition… has no validity and strikes every intelligent child as a juggle…for a point of space *is* a position, and can no more change its position than a leopard can change its spots." Heath (1925, v. 1, 225), recognizing that Euclid's "phraseology…leaves no room for doubt that he regarded one figure as actually *moved* and *placed upon* another," may well be right in concluding finally that Euclid must have inherited the approach and utilized it only "because he had not been able to see his way to a satisfactory substitute." Furthermore, because Euclid's articulation of 1.4 requires movement, scholars have taken pains to grapple with that movement, whether by recommending 1.4 as a definition (Peletier 1557, 16) or axiom (Russell 1903, 405) rather than a formal proposition, or by asserting the need for an additional postulate (Fitzpatrick, 2005, v. 1, p. 17, n. 6).

In fact, I propose, 1.4 can be proven using only those postulates, common notions, and propositions already available, though admittedly this method does require some adjustment to Euclid's own articulation of the proof itself, namely the (welcome) removal of any reference to *movement*. Here, using Fitzpatrick's 2005 translation of Heiberg's 1883 Greek text, is a brief illustration of how a portion of 1.4 may be expressed without movement, by causing a *third* line equal to a line of the first triangle to coincide with the corresponding equal line of the second triangle (strikethrough indicates material to be replaced by that which follows):

> ~~Let the triangle *ABC* be applied to the triangle *DEF*, the point *A* being placed on the point *D*, and the straight-line *AB* on *DE*.~~

> Let straight-line *EG* have been produced in a straight-line with *DE* [Post. 2], and let a straight-line DH, equal to the lesser straight-line *AB* be cut off from the greater straight-line *DG* [Prop. 3]. DH, being on the straight-line *DG*, will coincide with *DE*, on account of both *DH* and *DE* being equal to *AB* [C.N. 1].

Such a revised approach is of course meant not as a proposed emendation of Heiberg's text, which must stand as it is, but rather as an illustration of how 1.4 – whether even Euclid himself recognized it or not – can indeed be proven through the *Elements'* own Postulate 2, Common Notion 1, and Proposition 1.3. (A graphical illustration of the full Prop. 1.4 so revised will accompany the presentation.)

Bibliography

Fitzpatrick, R., trans. 2005. *Euclid's Elements in Greek: from Euclidis elementa*. 3 vols. Austin.

Heath, T. 1925. *The thirteen books of Euclid's Elements*. 2nd ed. 3 vols. Cambridge.

Heiberg, J. L., ed. 1883. *Euclidis elementa*. Leipzig.

Peletier, J. 1557. *In Euclidis Elementa geometrica demonstrationum libri sex*. Lyon.

Russell, B. 1903. *The Principles of Mathematics: vol. 1*. Cambridge.